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DISCONTINUOUS SOLUTIONS OF SEMILINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS PART II: p-CONSISTENCY

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DISCONTINUOUS SOLUTIONS OF SEMILINEAR DIFFERENTIAL-ALGEBRAIC EQUATIONS. PART II: $\mathcal{P}$-CONSISTENCY [1]

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ABSTRACT. Part I of this paper presented a theory of distribution solutions of semilinear differential-algebraic equations (DAE's). In particular, it was shown that uniqueness of solutions of initial value problems breaks down completely in the class of discontinuous solutions. Here a mathematical procedure is introduced for selecting physically acceptable solutions which satisfy some new consistency condition relative to admissible perturbations of the original DAE. Several nonlinear circuit examples are given to support the theory.

1. Introduction.

As shown in I, Section 3[3] of this paper, the difficulties associated with the existence of impasse points for DAE's of the form

$$A(t)\dot{x} = G(t,x) \tag{1.1}$$

can be partially resolved by considering distribution solutions instead of classical ones. Unfortunately, this creates the new problem of invalidating the uniqueness of solutions to initial value problems for (1.1) which, in general, will now have not one or even a few solutions, but uncountably many of them (see I, Example 3.1). Of course, this mathematical fact is hardly acceptable on physical grounds, and can only mean that most of the solutions, optimally all but one, are deprived of physical significance and should be discarded. This raises the question about suitable criteria for such a selection.

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[3] Throughout the presentation this notation refers to the corresponding item in Part I of this article.

At first sight, this task seems to require the specification of some physical setting for (1.1) and, in particular, of appropriate characteristics of the physically acceptable solutions. Here, we shall develop the thesis that a purely mathematical selection procedure can be introduced, and that physical data are needed only when it comes to the actual interpretation of the results. We begin with an outline of this idea.

Sometimes the opinion has been fostered that every DAE should or could be considered as the limiting case "$\epsilon = 0$" of a singularly perturbed ODE. However, physically motivated perturbations generally lead to such ODEs only after considerable manipulation of the equations and under global invertibility assumptions that are not necessarily satisfied by, or very difficult to check in, nontrivial concrete problems. In particular, the simple procedure of changing the algebraic constraints in (1.1) into differential equations weighed by the factor ϵ usually results in problems totally deprived of physical reality, and hence of no help for a better understanding of the case $\epsilon = 0$. In fact, what is often apparent is that the DAE (1.1) does represent a simplified model of some physical system in which small parameters have been neglected, but that the perturbed problems remain DAE's (and hence may be, though rarely are, explicit ODE's). Another difference is that the neglected parameters form a vector which reduces to a single scalar only in the simplest models.

Accordingly, we shall assume that (1.1) arises as a simplification of a perturbed DAE obtained by disregarding some physical parameters of negligible magnitude represented by a vector μ. Typically, the exact value of μ is unknown, but the μ-dependent perturbed problems, henceforth denoted by $\mathcal{P}_\mu$ where "$\mathcal{P}$" stands for "perturbation", can be written down explicitly since the nature of the neglected quantities is clear from the context. In this respect, see Remark 1.1 below.

With every fixed value of the small perturbation parameter μ we shall associate an open subset $W^{c,\mu}$ of the set of consistent points W^c of (1.1) (see Part I) depending only upon the direction of μ and consisting of points with the following property: If $(t_0, x_0) \in W^{c,\mu}$ then every solution of (1.1) passing through x_0 at time t_0 can be approximated uniformly in some open interval about t_0 by solutions of the perturbed problem $\mathcal{P}_{\epsilon\mu}$ as $\epsilon \to 0^+$. Thus, if μ is small in norm in the first place, solutions of (1.1) passing through points of $W^{c,\mu}$

are already closely approximated by solutions of $\mathcal{P}_{\epsilon\mu}$ when $\epsilon = 1$; that is, by solutions of $\mathcal{P}_\mu$. The set $W^{c,\mu}$ will be characterized in terms of an eigenvalue sign condition which also guarantees the aforementioned (local) approximation property via the Tikhonov-Levinson theorem of singular perturbation theory and its generalization to DAE's in the dissertation [Ya]. Points of $W^{c,\mu}$ will be called *$\mathcal{P}_\mu$-consistent points.*

Our criterion for the selection of the physically relevant solutions of (1.1) is based upon (i) the reasonable assumption that such solutions should be close (at least as distributions) to solutions of $\mathcal{P}_\mu$ if indeed $\mathcal{P}_\mu$ represents a realistic small perturbation of (1.1), and upon (ii) the postulate that physical solutions of (1.1) should not exhibit discontinuities (that is, they should be classical ones) unless their continuity creates a dichotomy with (i) above. This premise may perhaps be disputed in some problems, but we are not aware that its plausibility has thus far been challenged. In fact, it is implicitly made in all the traditional singular perturbation approaches.

The arguments developed so far provide an immediate justification to our procedure for selecting the solutions of (1.1) compatible with the perturbed problem $\mathcal{P}_\mu$. By definition, a solution of (1.1) will be called $\mathcal{P}_\mu$ - consistent if $(t, x(t)) \in W^{c,\mu}$ for all t in the time interval of interest, and if no jump occurs at any point of $W^{c,\mu}$.

It is important to notice that while the condition that no jumps occur at points of $W^{c,\mu}$ eliminates many otherwise possible distribution solutions, it does not rule out the possibility of jumps altogether, which may still exist when a solution reaches points of the boundary $\bar{W}^{c,\mu} \setminus W^{c,\mu}$. Furthermore, a jump will exist if, in addition, there exists a $\mathcal{P}_\mu$-consistent point where it terminates (otherwise, the solution is not extendable to larger time). In particular, jumps may always take place from singularities such as impasse points, which are not even consistent points (i.e. points of W^c). As we shall see, jumps are also possible from points which are not singular points of (1.1) and, in fact, which cannot be characterized independently of the perturbation.

Remark 1.1: For the claim that the solutions of (1.1) selected by the $\mathcal{P}_\mu$-consistency criterion are physically meaningful, it is of some importance that the components of the

vector μ account for *all* the physical parameters neglected in the original problem. The difficulty in re-introducing these parameters varies with the nature of the physical problem. For instance, in elementary particle mechanics, masses of rods (and sometimes of some particles) are commonly omitted from the models. In LRC networks, phenomena induced by the wiring are usually disregarded, and they can be accounted for by introducing parasitic series-inductances and shunt-capacitors. □

In the above approach the only obstacle to really singling out the perturbation of (1.1) with actual significance is the fact that the exact value of the parameter μ is not known in general. On the other hand, our examples indicate clearly that, *qualitatively*, this value is relatively unimportant, and that, in general, the solutions are observed to show only very few distinct behavior characteristics as μ is varied. This suggests that it may be possible to classify perturbed DAE problems in terms of inequivalent normal forms, in the very same spirit as has been done (to some degree) for perturbed bifurcation problems. A significant added difficulty is the necessarily global aspect of such perturbed DAE problems that jump phenomena make no sense in a framework of local analysis. These questions will not be considered further here.

Naturally, the idea of relating the solutions of DAE's to those of perturbed problems is not new. The eigenvalue condition central to this Part II has the visible appearance of a stability condition for equilibria. In the work on nonlinear circuits by Green and Willson [GW] stability is defined only for stationary points (operating points); that is, for solutions of the dc-circuit obtained by disregarding all dynamic elements. Then, in essence, an operating point is stable if under all possible perturbations by means of parasitic capacitors and inductances the corresponding stationary point of the resulting dynamic circuit is also stable.

On the other hand, Sastry and Desoer, [SDe], consider perturbations of a given circuit which produce a singularly perturbed ODE involving a scalar perturbation parameter. Then stability of a solution of the unperturbed problem can be defined by means of the conditions of the standard Tikhonov-Levinson theorem, which turn out to be closely related to those in this paper. One major difference here is that we allow perturbations to DAE

form and involving a vector of small parameters. Moreover, in our theory we consider, from the start, distribution solutions while, in [SDe], discontinuous solutions of (1.1) have to be defined as the limit of continuous solutions of the perturbed problems. Thus, in our approach, the solutions of (1.1) are defined independently of the perturbations which become involved only later, in the selection process. A major technical advantage is that we do not require any assumption about the behavior of the solutions of the perturbed problems, and thus can bypass serious difficulties in the vicinity of jumps. As a result, we are able to handle situations both frequently encountered and not covered by the analysis of [SDe]. On the other hand, the issue whether our $\mathcal{P}_\mu$-consistent solutions can be approximated in the sense of distributions (and hence *globally*) by the solutions of the problems $\mathcal{P}_{\epsilon\mu}$ as $\epsilon \to 0^+$ is an important question as yet unresolved by the segment of the theory presented in this paper.

Section 2 provides the technical details of the preceding discussion. The relationship of $\mathcal{P}$-consistency; that is, $\mathcal{P}_\mu$ - consistency for some μ, with the Tikhonov-Levinson theorem is examined in Section 3. Our aim there is to justify the choice of our conditions by showing how they relate to this classical result of singular perturbation theory. It is a significant advantage of our method that the extremely cumbersome technicalities involved in this process never have to be worked out in any concrete example. In contrast, any approach based upon eventual reduction to a standard perturbation problem for ODEs, when possible at all, would have to display similar technicalities in every case.

In Section 4, several examples from electrical circuit theory are given. As already mentioned, in such problems the natural perturbations are represented by parasitic inductances and capacitors, which turns out to be in concordance with Smale's proposal in [Sm].

The notation and terminology are those of Part I, to which the reader is referred regarding all symbols and concepts not defined here.

2. $\mathcal{P}$-consistency.

As noted in the Introduction, our paradigm is that, in practice, many DAE's (1.1) arise

"naturally" as limiting cases of perturbed DAE's of the form

$$A(\mu,t)\dot{x} = G(\mu,t,x), \tag{2.1}$$

such that the DAE (1.1) is recovered for $\mu = 0$. If so, $A(t)$ and $G(t,x)$ in (1.1) become $A(0,t)$ and $G(0,t,x)$, respectively. Above, by "naturally" we mean that the vector $\mu \in \mathbb{R}^k$ represents a collection $(\mu_1, \cdots, \mu_k)$ of negligible physical constants. An apparently innocuous fact will turn out to have a crucial importance for the justification of our arguments: In most, if not all cases, physical constants have meaning only for *positive* values (after possible rescaling and sign changes). As a result, the perturbations (2.1) involve only vectors μ with positive components. Accordingly, we shall set

$$\mathcal{C}_+ = \{\mu \in \mathbb{R}^k_+ : \|\mu\| <\}, \tag{2.2}$$

where $\mathbb{R}_+ = (0,\infty)$ and k is a nonnegative integer (when $k = 0$, no physical constant has been neglected in (1.1); this case does fall within the framework of this paper and is discussed in Remark 2.4 later). In (2.2), the normalization $\|\mu\| < 1$ could be replaced by $\|\mu\| < \eta$ where $\eta > 0$ is arbitrary. The choice $\eta = 1$ is made here for convenience, and mainly to avoid introducing a parameter η which would never be involved in further considerations.

Remark 2.1: In (2.2), the choice $\mu \in \mathbb{R}^k_+$ is justified because it is assumed that μ represents *only* physical parameters which are *ignored* in (1.1) (and hence set to 0 there). This is an important point, for if μ includes components representing perturbations of physical constants already present in (1.1), then negative as well as positive values of these components are perfectly admissible. There is no difficulty to incorporate perturbations of already existing physical parameters, thus letting (2.1) depend upon an additional vector ν. Under conditions frequently met in applications, doing so does not modify the final results, and the behavior of interest continues to be solely the consequence of those parameters that are not already embodied in (1.1). Thus, to keep technical matters as simple as possible and not to obscure the issue, this otherwise legitimate generalization will not be considered here. □

Standard perturbation theory uses $k = 1$ in (2.2) and, hence, a positive, real parameter μ (usually called ϵ). Moreover, it requires that (2.1) is an ODE; that is, $A(\mu, t)$ is invertible for any vector μ with positive, and sufficiently small components.

For technical reasons, we assume that the mappings

$$(\mu, t) \in \bar{\mathcal{C}}_+ \times J \longmapsto A(\mu, t) \in \mathcal{L}(\mathbb{R}^n) \tag{2.3}$$

and

$$(\mu, t, x) \in \bar{\mathcal{C}}_+ \times J \times \mathbb{R}^n \longmapsto G(\mu, t, x) \in \mathbb{R}^n, \tag{2.4}$$

are of class C^∞ (sufficient smoothness would suffice) and that

$$\text{rank } A(\mu, t) = \rho \geq r = \text{ rank } A(0, t), \quad \forall (\mu, t) \in \mathcal{C}_+ \times J \tag{2.5}$$

and

$$\ker A(\mu, t) \subset \ker A(0, t), \quad \forall (\mu, t) \in \mathcal{C}_+ \times J. \tag{2.6}$$

Thus ρ is independent of (μ, t), while the independence of r from t was already assumed in Part I. The condition (2.6) can be weakened; see Remark 2.6 below.

Evidently, (2.1) is an ODE for $\mu \in \mathcal{C}_+$ if and only if $\rho = n$ in (2.5), so that (2.6) becomes vacuous. For future use, we note that (2.6) implies that

$$\dim \left(\ker \frac{\partial A}{\partial \mu}(0, t) \right)_{|_{\ker A(0,t)}} \geq n - \rho, \quad \forall (\mu, t) \in \mathcal{C}_+ \times J. \tag{2.7}$$

Indeed, if $t \in J$ is fixed and (ϵ_j) is a sequence of positive numbers tending to 0, and $h_{j,1} \cdots, h_{j,n-\rho}$ denotes an orthonormal basis of $\ker A(\epsilon_j \mu, t)$, then we have $h_{j,i} \in \ker A(0, t)$ and hence

$$\frac{1}{\epsilon_j}(A(\epsilon_j \mu, t) - A(0, t))h_{j,i} = 0, \quad 1 \leq i \leq n - \rho, \ j > 0. \tag{2.8}$$

After extracting a subsequence, we may assume that $\lim_{j \to \infty} h_{j,i} = h_i, 1 \leq i \leq n - \rho$, so that the vectors $h_1, \cdots, h_{n-\rho}$ form an orthonormal subset of $\ker A(0, t)$. Now, by taking the limit in (2.8), we find that $((\partial A/\partial \mu)(0, t)\mu)h_i = 0, 1 \leq i \leq n - \rho$, and (2.7) follows.

As in Part I, we focus only on the case when, for $\mu = 0$, (2.1) (that is, (1.1)) is a geometrically nonsingular DAE of index one. Let $(t,x) \in W^c$ be a consistent point for (1.1), so that $Q(t)D_xG(0,t,x)|_{\ker A(0,t)} \in GL(\ker A(0,t), Z)$ by I, Proposition 2.1, where $Q(t)$ is the projection used in I, Section 2 and corresponding to $A(t) = A(0,t)$. (In other words, $Q(t)$ projects onto the fixed complement Z of rge $A(0,t), t \in J$). For notational convenience, we set

$$\Lambda(\mu,t,x) = \left[Q(t)D_xG(0,t,x)|_{\ker A(0,t)}\right]^{-1} Q(t)\left(\frac{\partial A}{\partial \mu}(0,t)\mu\right) \in \mathcal{L}(\ker A(0,t)). \tag{2.9}$$

Definition 2.1. *For fixed $\mu \in \mathcal{C}_+$ the point $(t,x) \in W^c$ is consistent with the perturbation (2.1), or $\mathcal{P}_\mu$-consistent for short, if the operator $\Lambda(\mu,t,x)$ in (2.9) has $\rho - r$ eigenvalues (counting multiplicities) with strictly negative real part. The set of $\mathcal{P}_\mu$-consistent points will be denoted by $W^{c,\mu}$.*

Occasionally, we shall refer to a point (t,x) being $\mathcal{P}$-consistent if it is $\mathcal{P}_\mu$-consistent for some $\mu \in \mathcal{P}$. This terminology is compatible with calling (2.1) for fixed $\mu \in \mathcal{C}_+$ the "perturbed problem $\mathcal{P}_\mu$" and calling $\mathcal{P}$ the collection of perturbed problems $\mathcal{P}_\mu$ for $\mu \in \mathcal{C}_+$. The notation also reflects the obvious fact that, not only what value the parameter μ has, but also, and primarily, how μ enters the system (2.1) is important here.

By (2.7), the operator $\Lambda(\mu,t,x)$ in (2.9) has 0 as an eigenvalue with geometric multiplicity at least equal to $n - \rho$. Hence, Definition 2.1 applies if and only if 0 is an eigenvalue of algebraic and geometric multiplicity exactly equal to $n - \rho$ while its other eigenvalues have negative real parts. Note also that the condition of Definition 2.1 is valid if and only if it holds with μ replaced by any positive scalar multiple. On the other hand, because of $\Lambda(-\mu,t,x) = -\Lambda(\mu,t,x)$, Definition 2.1 would be meaningless if μ and $-\mu$ could both represent admissible perturbations, unless $\rho = r$ and hence $\Lambda \equiv 0$ in (2.9). This is ruled out by the condition $\mu \in K$ which, in turn, was earlier dictated by the positivity of physical constants.

Remark 2.2: For given $u \in \ker A(0,t)$ we see that $v := \Lambda(\mu,t,x)\ u \in \ker A(0,t)$ is

characterized by the condition

$$\left(\frac{\partial A}{\partial \mu}(0,t)\mu\right)u - D_x G(0,t,x)v \in \text{ rge } A(t),$$

which is clearly equivalent to (2.9). This shows that $\Lambda(\mu,t,x)$, and hence its eigenvalues and their multiplicities, are independent of the choice of $Q(t)$. □

Proposition 2.1. *For $\mu \in K$, the set $W^{c,\mu}$ is open in W^c.*

Proof. Suppose this does not hold. Then, there exists a $(t_*,x_*) \in W^{c,\mu}$ and a sequence $(t_j,x_j) \in W^c \setminus W^{c,\mu}$ with $\lim_{j\to\infty}(t_j,x_j) = (t_*,x_*)$. Since (t_j,x_j) is not $\mathcal{P}_\mu$-consistent, the operator $\Lambda(\mu,t_j,x_j)$ has at least $n-\rho+1$ eigenvalues (counting multiplicities) with nonnegative real part. In essence, a contradiction is reached by using the continuity of the eigenvalues with respect to the operator (see [K]), which implies that $\Lambda(\mu,t_*,x_*)$ has at least $n-\rho+1$ eigenvalues with nonnegative real parts, whence $(t_*,x_*) \notin W^{c,\mu}$.

The above argument requires the domain of the operators to be the same, which here is not the case since $\ker A(0,t)$ varies with t. This apparent difficulty is overcome by considering a one-to-one mapping $U(t) \in \mathcal{L}(\mathbb{R}^{n-r},\mathbb{R}^n)$ with range $\ker A(0,t)$ which depends continuously upon t near t_*. Existence of U is easily derived from the constancy of $\dim\ker A(0,t)(=n-r)$. The eigenvalues of $\Lambda(\mu,t,x)$ for (t,x) near (t_*,x_*) are also those of

$$[Q(t)D_xG(0,t,x)U(t)]^{-1}Q(t)\left(\frac{\partial A}{\partial \mu}(0,t,\mu)\right)U(t) \in \mathcal{L}(\mathbb{R}^{n-r}),$$

and the continuity argument may now be used with these operators. This completes the proof. □

The notion of $\mathcal{P}_\mu$-consistent point leads at once to that of $\mathcal{P}_\mu$-consistent solution:

Definition 2.3. *Let $\mu \in \mathcal{C}_+$ be given. The function $x : J \to \mathbb{R}^n$ is a $\mathcal{P}_\mu$ – consistent solution of (1.1) if*

(i) $(t,x(t)) \in W^c \Rightarrow (t,x(t)) \in W^{c,\mu}$.

(ii) For $t_0 \in J$ and $(t_0,x(t_0)) \in W^{c,\mu}$, there is a $T>0$ such that $x_{|[t_0,t_0+T)}$ coincides with the unique classical solution of (1.1) passing through $x(t_0)$ for $t=t_0$ (see I, Theorem 2.1).

The union of all the $\mathcal{P}_\mu$-consistent solutions, as $\mu \in \mathcal{C}_+$ is varied, is called the set of $\mathcal{P}$-consistent solutions.

By Definition 2.2, a $\mathcal{P}_\mu$-admissible solution x of (1.1) cannot jump from points of $W^{c,\mu}$. Nevertheless, it may have a discontinuity at a point $(t_0, x_0^-) \in \bar{W}^{c,\mu} \setminus W^{c,\mu}$ if $(t, x(t)) \in W^{c,\mu}$ for $t < t_0$ and $\lim_{t \to t_0^-} x(t) = x_0^-$. This may happen when $x_0^- \in W^c$ (see Remark 2.3 later and Example 3 of Section 4), or when x_0^- is an accessible impasse point (see I, Theorems 3.1 and 3.2) provided that there is a $(t_0, x_0^+) \in W^{c,\mu}$ such that $x_0^+ - x_0^- \in \ker A(0, t_0)$. Jumps may also occur at other singularities, but then it must be proved that the discontinuous function so obtained solves (1.1) in the sense of distributions. This amounts to justifying the validity of formula I (3.1) which, as noted in Part I, should not be taken for granted.

Because $(t, x(t))$ must lie entirely in $W^{c,\mu}$ when x is a $\mathcal{P}_\mu$-consistent solution of (1.1), the occurrence of discontinuities is considerably restricted. However, this does not necessarily reinstate the uniqueness of solutions of the initial value problems. In fact, for a point $(t_0, x_0^-) \in \bar{W}^{c,\mu} \setminus W^{c,\mu}$, the "new" starting point x_0^+ need not be uniquely determined in $W^{c,\mu}$. As noted at the end of Part I, there will be, in general, only a finite, or at most countable, number of possible choices for x_0^+. Of course, it may also happen that no choice for x_0^+ is available, and hence the solution $x(t)$ stops at x_0^- and cannot be continued further.

We note also that because of the openness of the set $W^{c,\mu}$ (Proposition 2.1) along with condition (ii) of Definition 2.2, $\mathcal{P}_\mu$ - consistent solutions of (1.1) can exhibit jumps only at isolated (and hence discrete) time values. Thus, altogether, we find that, in general, there are at most countably many (and often only finitely many) $\mathcal{P}_\mu$-consistent solutions of (1.1) with prescribed initial value defined in the open interval J.

Remark 2.3: From the above comments, $\mathcal{P}_\mu$ - consistent solutions of (1.1) are solutions which are classical ones unless they have no other choice. Reasons for exhibiting discontinuities are either intrinsic (singularities) or perturbation-dependent (two conjugate eigenvalues of the operator $\Lambda(\mu, t, x)$ in (2.9) cross the imaginary axis in the "wrong" direction; that is, pass from negative to positive real part). In both cases, jumps occur from

the impossibility of approximating would-be classical solutions of (1.1) by classical solutions of $\mathcal{P}_{\epsilon\mu}$ (i.e. (2.1) with μ replaced by $\epsilon\mu$) as $\epsilon \to 0^+$ in the vicinity of the point (t_0, x_0^-) where the jump originates. The relationship between $\mathcal{P}_\mu$-consistency and approximation will be investigated in detail in the next section. For the time being, note only that a solution $x(t)$ of (1.1) can certainly not be approximated by any continuous function in any open interval centered at t_0 when $\lim_{t \to t_0^-} x(t) = x_0^-$ and (t_0, x_0^-) is an (accessible) impasse point of (1.1). Indeed, in this case, there is *no* continuous extension of $x(t)$ as a classical solution of (1.1) beyond $t = t_0$. □

Remark 2.4: It is not without interest to apply this theory to the case when $\rho = r$ in (2.5). If so, we have $\Lambda(\mu, t, x) = 0$ in (2.9) (see (2.7)) for every $(t, x) \in W^c$, and hence $\Lambda(\mu, t, x)$ has $0 = \rho - r$ eigenvalues with negative real part. In other words, $W^c = W^{c,\mu}$ for all μ. As a result, jumps may still occur, but only at singularities (impasse points in particular) which of course need not always exist (as, e.g., for monotone networks; see [DeW] and the references therein). This case is relevant when (1.1) is replaced by (2.1) for fixed $\mu = \mu_0$ and (2.1) already contains "all" the possible perturbations of (1.1) (see Remark 1.1), in other words, when no physical constant is neglected in $\mathcal{P}_{\mu_0}$. Note that the problems $\mathcal{P}_\mu$ for μ close to μ_0 are then perturbations of $\mathcal{P}_{\mu_0}$ of the kind alluded to in Remark 2.1; that is, those that do not involve any physical parameter not already represented in $\mathcal{P}_{\mu_0}$. □

An important aspect of the concept of $\mathcal{P}$-consistent solution is its *strong bias toward evolution in positive time*. Indeed, the concept of $\mathcal{P}_\mu$-consistent point is not invariant under time-reversal because the change of variable $t = T - s, x(t) = y(T - s)$ transforms (2.1) into the system

$$A(\mu, T - s)\frac{dy}{ds} = -G(\mu, T - s, y), \tag{2.10}$$

and hence (1.1) (with $A(t) = A(0, t), G(t, x) = G(0, t, x)$) into

$$A(T - s)\frac{dy}{ds} = -G(T - s, y). \tag{2.11}$$

For (2.11) the manifolds W and W^c are obtained from the corresponding manifolds W and W^c for (1.1) via the variable change $s = T - t$. Such a simple correspondence does not carry

over to the sets of $\mathcal{P}_\mu$-consistent points. In fact, the operator $\Lambda(\mu, t, x)$ in (2.9) is changed into $-\Lambda(\mu, T-s, x)$ (because of the change of G into $-G$ in (2.11)) and hence, if (t, x) is $\mathcal{P}_\mu$-consistent for (1.1), then $(T-t, x)$ will *never* be consistent with the perturbation (2.10) of (2.11), unless $\rho = r$ in (2.5).

For problems in which t represents physically the time variable, this is unimportant since physical relevance goes along with evolution in positive time. But even in problems where the variable t may increase or decrease, e.g., a load parameter, the above remark is not necessarily a sign of the inadequacy of the concept of $\mathcal{P}$-consistent solution. Rather, it indicates that the same perturbations (2.1) cannot be used to describe the physical phenomena of interest when t both increases as well as decreases. This claim can be corroborated for elementary one-dimensional plasticitiy models where, indeed, some sign change in part of the equations must occur upon switching from loading to unloading to preserve physical correctness.

Remark 2.5: (Inconsistent initial conditions): The concept of $\mathcal{P}$-consistency can be used in connection with the problems of inconsistent initial conditions discussed in I, Section 4. In fact, the discussion there can be repeated almost verbatim when the initial condition x_0^- is not $\mathcal{P}_\mu$-consistent. In this case, x_0^- should be replaced by a point x_0^+ such that $x_0^+ - x_0^- \in \ker A(t_0)$ and (t_0, x_0^+) is $\mathcal{P}_\mu$-consistent. Clearly, the $\mathcal{P}_\mu$ - consistency requirement places a further limitation upon the admissible values x_0^+. Dually, since lack of $\mathcal{P}_\mu$ - consistency is more frequent than lack of consistency in the sense of Part I, there are now more points x_0^- from which an initial jump should be expected. □

We complete this section with the observation that $\mathcal{P}$ - consistency has the desirable property of being invariant under changes of variables preserving the semilinear structure of the system (2.1). In other words, if the system (2.1) is multiplied by $N(\mu, t)$ and x is changed into $M(\mu, t)y$ with smooth (or sufficiently smooth) $N(\mu, t), M(\mu, t) \in GL(\mathbb{R}^n)$, thus becoming a transformed system $\tilde{\mathcal{P}}_\mu$, then the $\mathcal{P}_\mu$-consistent points (resp. solutions) of (1.1) are converted into the $\tilde{\mathcal{P}}_\mu$-consistent points (resp. solutions) of the unperturbed transformed system (and vice-versa). The verification is straightforward using Remark 2.1 and will not be given here.

Remark 2.6: It is noteworthy that the eigenvalue condition for the operator $\Lambda(\mu,t,x)$ is preserved even when the operator $A(\mu,t)M(\mu,t)$ does *not* satisfy the condition (2.6). This can be used in reverse to extend the validity of the theory presented here to the case when (2.6) is replaced by the weaker requirement that

$$\ker A(\mu,t)M(\mu,t) \subset \ker A(0,t)M(0,t), \quad \forall (\mu,t) \in \mathcal{C}_+ \times J,$$

for some smooth enough mapping $M : \bar{\mathcal{C}}_+ \times J \to GL(\mathbb{R}^n)$. □

3. Relationship with the Tikhonov-Levinson Theorem.

In order to justify the definitions of the $\mathcal{P}$-consistent points and of the $\mathcal{P}$-consistent solutions of (1.1) by making the connection with the local approximation property mentioned in the Introduction, we first examine the particular case when $\rho = n$ in (2.5) and $A(\mu,t)$ has the special form

$$A(\mu,t) = \begin{pmatrix} I_r & 0 \\ 0 & B(\mu,t) \end{pmatrix}, \tag{3.1}$$

where $B(0,t) \equiv 0$ and $B(\mu,t) \in \mathcal{L}(\mathbb{R}^{n-r})$. The system (2.1) now has the form

$$\begin{cases} \dot{x}_1 = g_1(\mu,t,x), \\ B(\mu,t)\dot{x}_2 = g_2(\mu,t,x). \end{cases} \tag{3.2}$$

The manifold W is the zero set of the mapping $g_2(0,\cdot,\cdot)$, and a point (t,x) is in W^c if and only if

$$g_2(0,t,x) = 0, \quad \frac{\partial g_2}{\partial t}(0,t,x) - \frac{\partial g_2}{\partial x_1}(0,t,x)g_1(0,t,x) \in \text{ rge } \frac{\partial g_2}{\partial x_2}(0,t,x). \tag{3.3}$$

By I, Proposition 2.1, (1.1) is geometrically nonsingular of index one if and only if (3.3) is equivalent to

$$g_2(0,t,x) = 0, \quad \frac{\partial g_2}{\partial x_2}(0,t,x) \in GL(\mathbb{R}^{n-r}). \tag{3.4}$$

Thus, for a given $\mu \in K$ as in Definition 2.1, we find that $(t,x) \in W^{c,\mu}$ if (3.4) holds and the operator

$$\frac{\partial g_2}{\partial x_2}(0,t,x)^{-1}\left(\frac{\partial B}{\partial \mu}(0,t)\mu\right) \in \mathcal{L}(\mathbb{R}^{n-r}) \tag{3.5}$$

has $n-r$ eigenvalues with negative real part. If so, the operator (3.5) is invertible and, of course, its inverse

$$\left(\frac{\partial B}{\partial \mu}(0,t)\mu\right)^{-1}\frac{\partial g_2}{\partial x_2}(0,t,x)\in\mathcal{L}(\mathbb{R}^{n-r}) \tag{3.6}$$

has $n-r$ eigenvalues with negative real part.

Given $\epsilon>0$, and because of $B(0,t)=0$, we have $B(\epsilon\mu,t)=\epsilon\tilde{B}(\epsilon,\mu,t)$ with $\tilde{B}(\epsilon,\mu,t)=\int_0^1\frac{\partial B}{\partial\mu}(s\epsilon\mu,t)\mu ds$. In particular, $\tilde{B}(0,\mu,t_0)=\frac{\partial B}{\partial\mu}(0,t_0)\mu$, and hence $\tilde{B}(\epsilon,\mu,t)$ is invertible for sufficiently small $\epsilon>0$ and t near t_0. Thus, for those values of ϵ and t, the system (3.2) with μ replaced by $\epsilon\mu$ is equivalent to

$$\begin{cases}\dot{x}_1=g_1(\epsilon\mu,t,x)\\ \epsilon\dot{x}_2=\tilde{B}(\epsilon,\mu,t)^{-1}g_2(\epsilon\mu,t,x)\end{cases} \tag{3.7}$$

The key point now is the following: Let $x(t)$ denote the unique solution of (3.7) for $\epsilon=0$ (i.e. of (3.2) with $\mu=0$) satisfying $x(t_0)=x_0$, where (t_0,x_0) is such that (3.4) holds. Then the main hypothesis of the Tikhonov-Levinson theorem (see e.g. the references in [H] or [OM]), which ensures that $x(t)$ can be uniformly approximated by solutions of (3.7) with $\epsilon>0$ in some sufficiently small interval about t_0, is that the operator $\tilde{B}(0,\mu,t_0)^{-1}\frac{\partial g}{\partial x_2}(0,t_0,x_0)$ has $n-r$ eigenvalues with negative real part. Since $\tilde{B}(0,\mu,t_0)=\frac{\partial B}{\partial\mu}(0,t_0)\mu$, this is precisely our condition for the operator (3.6). This justifies the "approximation" statement made earlier in Remark 2.2, at least when $A(\mu,t)$ has the form (3.1) and $\rho=n$ in (2.5).

We now justify the relevance of the $\mathcal{P}$-consistent solutions without assuming the special form (3.1). But we do continue (momentarily) to require that (2.1) is an ODE. Our argument here is simply that, locally in (μ,t), reduction to the form (3.1) is always possible. We skip a number of straightforward though tedious technicalities and focus on the main points. The desired reduction can be accomplished via transformations

$$U(\mu,t)\in GL(\mathbb{R}^n),\quad V(\mu,t)\in GL(\mathbb{R}^n)$$

such that $V(\mu,t)A(\mu,t)U(\mu,t)$ has the form (3.1), followed by introducing the change of variables $x(t)=U(\mu,t)y(t)$. In this way, the right-hand side $G(\mu,t,x)$ becomes

$$V(\mu,t)[-A(\mu,t)\dot{U}(\mu,t)y+G(\mu,t,U(\mu,t)y)],$$

and the projection $Q(t)$ may be replaced by the projection $V(0,t)Q(t)V(0,t)^{-1}$. The old and new manifolds W and W^c are related via $(t,x) = (t, U(0,t)y)$, and, by using $Q(t)A(0,t) = 0$ and

$$\ker V(0,t)A(0,t)U(0,t) = U(0,t)^{-1} \ker A(0,t),$$

it is easily seen that the operator $Q(t)D_xG(0,t,x)|_{\ker A(0,t)}$ becomes

$$V(0,t)Q(t)D_xG(0,t,U(0,t)y)U(0,t)|_{U(0,t)^{-1}\ker A(0,t)}. \tag{3.8}$$

Finally, $Q(t)\left(\frac{\partial A}{\partial \mu}(0,t)\mu\right)\Big|_{\ker A(0,t)}$ is transformed into

$$V(0,t)Q(t)V(0,t)^{-1}\left(\frac{\partial VAU}{\partial \mu}(0,t)\mu\right)\Big|_{U(0,t)^{-1}\ker A(0,t)}. \tag{3.9}$$

This can be simplified by noting that two of the three terms arising in the expansion of $\partial(VAU)/\partial\mu$ yield no contribution. Indeed, we have

$$V(0,t)Q(t)V(0,t)^{-1}\left(\frac{\partial V}{\partial \mu}(0,t)\mu\right)A(0,t)U(0,t)|_{U(0,t)^{-1}\ker A(0,t)} = 0,$$

because of $A(0,t)|_{\ker A(0,t)} = 0$, and

$$V(0,t)Q(t)V(0,t)^{-1}V(0,t)A(0,t)\left(\frac{\partial U}{\partial \mu}(0,t)\mu\right)\Big|_{U(0,t)^{-1}\ker A(0,t)} = 0,$$

because of $Q(t)A(0,t) = 0$. Thus, (3.9) reads

$$V(0,t)Q(t)\left(\frac{\partial A}{\partial \mu}(0,t)\mu\right)U(0,t)|_{\ker U(0,t)^{-1}A(0,t)},$$

and, together with (3.8), this shows that the "new" operator $\Lambda(\mu,t,x)$ is simply

$$U(0,t)^{-1}\Lambda(\mu,t,U(0,t)y)U(0,t)$$

Since $x = U(0,t)y$, this operator has the same eigenvalues as $\Lambda(\mu,t,x)$ in (2.9).

This shows that the transformation $x = U(0,t)y$ also relates the old and new $\mathcal{P}_\mu$-consistent points, and hence the transformation $x(t) = U(0,t)y(t)$ relates the old and

new $\mathcal{P}_\mu$-consistent solutions. Since the solutions of the old and new perturbed problems are in correspondence through $x(t) = U(\mu,t)y(t)$, the relationship between $\mathcal{P}_\mu$-consistent solutions and solutions of the perturbed problems is the same before and after reduction of (2.1) to the form (3.1).

Finally, when (2.1) is a (completely nonsingular) DAE for $\mu \in \mathbb{R}^k_+$, a similar reduction procedure changes the problem into one in which $A(\mu,t)$ takes the form

$$A(\mu,t) = \begin{pmatrix} I_r & 0 & 0 \\ 0 & B(\mu,t) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where now $B(\mu,t) \in \mathcal{L}(\mathbb{R}^{\rho-r})$ and $B(0,t) \equiv 0$. Once again, the transformation preserves $\mathcal{P}_\mu$-consistent points. The transformed system (2.1) has the form

$$\begin{cases} \dot{x}_1 = g_1(\mu,t,x) \in \mathbb{R}^r, \\ B(\mu,t)\dot{x}_{21} = g_{21}(\mu,t,x) \in \mathbb{R}^{\rho-r} \\ 0 = g_{22}(\mu,t,x) \in \mathbb{R}^{n-\rho}, \end{cases} \tag{3.10}$$

where we have set $x = (x_1, x_2)$ with $x_2 = (x_{21}, x_{22})$ and $g = (g_1, g_2)$ with $g_2 = (g_{21}, g_{22})$. Points of W continue to be characterized by (3.3), and geometric nonsingularity still amounts to equivalence between (3.3) and (3.4). Thus, $(\partial g_2/\partial x_2)(0,t,x)$ is invertible at each point $(t,x) \in W^c$. Such a point is $\mathcal{P}_\mu$-consistent if and only if the operator

$$\frac{\partial g_2}{\partial x_2}(0,t,x)^{-1} \begin{pmatrix} \frac{\partial B}{\partial \mu}(0,t)\mu & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{n-r}) \tag{3.11}$$

has $\rho - r$ eigenvalues with negative real part for $\mu \in \mathcal{P}$, which, in particular, implies that $\frac{\partial B}{\partial \mu}(0,t)\mu$ is invertible. The eigenvalues of (3.11) are the numbers $\lambda \in \mathbb{C}$ for which there is $h_2 \in \mathbb{C}^{n-r} \setminus \{0\}$, $h_2 = (h_{21}, h_{21})$ such that

$$\lambda \begin{pmatrix} \frac{\partial g_{21}}{\partial x_{21}}(0,t,x) & \frac{\partial g_{21}}{\partial x_{22}}(0,t,x) \\ \frac{\partial g_{22}}{\partial x_{21}}(0,t,x) & \frac{\partial g_{22}}{\partial x_{22}}(0,t,x) \end{pmatrix} \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial B}{\partial \mu}(0,t)\mu\right) h_{21} \\ 0 \end{pmatrix}$$

Assuming that $(\partial g_{22}/\partial x_{22})(0,t,x)$ is invertible, we conclude that the nonzero eigenvalues λ are characterized by the property that $1/\lambda$ is an eigenvalue of the operator

$$\left(\frac{\partial B}{\partial \mu}(0,t)\mu\right)^{-1} \left[\frac{\partial g_{21}}{\partial x_{21}} - \frac{\partial g_{21}}{\partial x_{22}}\left(\frac{\partial g_{22}}{\partial x_{22}}\right)^{-1} \frac{\partial g_{22}}{\partial x_{21}}\right](0,t,x) \in \mathcal{L}(\mathbb{R}^{\rho-r}). \tag{3.12}$$

The condition that the operator (3.12) has $\rho - r$ eigenvalues with negative real part at some point $(t_0, x_0) \in W^c$ is exactly the (main) condition required in the dissertation [Y] for the uniform approximation, in the vicinity of t_0, of the solution $x(t)$ of the problem (3.10) with $\mu = 0$, satisfying $x(t_0) = x_0$, by solutions of (3.10) with μ replaced by $\epsilon\mu, \epsilon > 0$. Thus, once again, our characterization of $\mathcal{P}$-consistent points and definition of $\mathcal{P}$-consistent solutions relates very closely to (a generalization of) the Tikhonov-Levinson theorem.

Remark 3.1: Consistent with the nature of the arguments presented in this section, it is quite natural to ask when a $\mathcal{P}_\mu$-consistent solution of (1.1) is the limit of (classical) solutions of (2.1) with μ replaced by $\epsilon\mu, \epsilon > 0$, in the sense of distributions. We do not have a general answer to this important question. Note, however, that the addition of this (global) requirement for the admissibility of solutions of (1.1) can only confine further the set of $\mathcal{P}_\mu$-consistent solutions. As observed in Section 2, this set is in general already countable, and often finite, when an initial condition is prescribed. This is to say that, at this stage, there is only a small number of candidates left that may, or may not, pass the aforementioned global approximation test. (The difficulty in the global approximation question is as follows: unlike the continuous solutions of $\mathcal{P}_\mu$, the discontinuous solutions of (1.1) are insensitive to the driving vector field between the initial and terminal points of the jump. Thus, solutions of (1.1) may sometimes be continued by jumping over attractors which otherwise capture the solutions of $\mathcal{P}_\mu$.) □

4. Examples from Nonlinear Circuit Theory.

Example 1: Our first example is the simple, well-studied circuit (see e.g. [CD] and [SDe]) of Figure 4.1

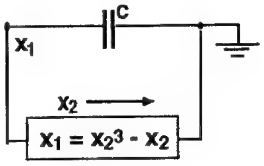

Figure 4.1

which is modelled by the autonomous DAE

$$
\begin{aligned}
C\dot{x}_1 &= -x_2, \\
0 &= x_1 - x_2^3 + x_2.
\end{aligned}
\tag{4.1}
$$

The solutions are constrained to the cubic path $\{x \in \mathbb{R}^2 : x_1 = x_2^3 - x_2\}$ of Figure 4.2. It is trivial to check that (4.1/2) is geometrically nonsingular of index one. In line with I, Remark 2.4 (change of notation for autonomous problems), the points $\xi_1 = (-2/(3\sqrt{3}), 1/\sqrt{3})^T$, and $\xi_2 = -\xi_1$ are accessible impasse points. As discussed, e.g., in [CD], or [SDe], at these points the solution is certainly expected to exhibit jump phenomena.

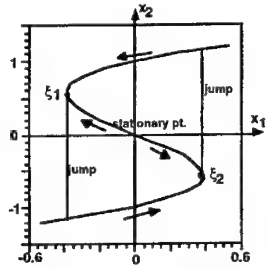

Figure 4.2

For the circuit of Figure 4.1 it is well-known (see loc. cit.) that a small parasitic line inductance $\mu > 0$ in series with the nonlinear resistor should not be disregarded. The resulting perturbed circuit in Figure 4.3 is modelled by the DAE

$$
\begin{aligned}
C\dot{x}_1 &= -x_2, \\
\mu\dot{x}_2 &= x_1 - x_3, \\
0 &= x_3 - x_2^3 + x_2.
\end{aligned}
\tag{4.2}
$$

Evidently, for $\mu = 0$ this becomes an augmented version of (4.1) in which the trivial equation $0 = x_1 - x_3$ has been added and the variable x_1 in the last equation of (4.1) has been replaced by x_3. It should be evident that (4.1) and its augmented form are fully equivalent except that the solutions are now constrained to the cubic path $W = \{x \in \mathbb{R}^3 :$

$x_1 = x_3,\ x_3 = x_2^3 - x_2\}$ in $\mathbb{R}^3$. The points ξ_1 and ξ_2 become the points $\tilde{\xi}_1$ and $\tilde{\xi}_2$ of W, where $\tilde{\xi}_1 = (-2/3\sqrt{3}, 1/\sqrt{3}, -2/3\sqrt{3})^T$ and $\tilde{\xi}_2 = -\tilde{\xi}_1$.

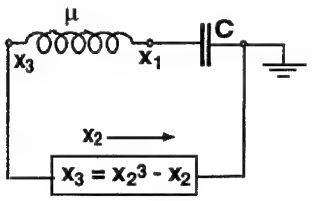

Figure 4.3

Again, in line with I, Remark 2.4 it is readily checked that, for $\mu > 0$, we have $W^c(\mu) = \{x \in \mathbb{R}^3 : x_3 - x_2^3 + x_2 = 0\}$ and that (4.2) is geometrically nonsingular. The operator Λ of (2.9) has here the matrix representation

$$\Lambda(\mu, x) = \begin{pmatrix} \frac{\mu}{1-3x_2^2} & 0 \\ -\mu & 0 \end{pmatrix}$$

Thus, for $\mu \in \mathcal{C}_+ = (0, 1)$ (see (2.2)), the set of $\mathcal{P}_\mu$-consistent points is

$$W^{c,\mu} = \{x \in W : |x_2| > \frac{1}{\sqrt{3}}\}.$$

This excludes jumps from any point on the upper or lower branches of the S-shaped curve W except, of course, $\tilde{\xi}_1$ or $\tilde{\xi}_2$. From, say, $\tilde{\xi}_1$ the jump has to end at a point $x^+ \in W$ such that $x^+ - \tilde{\xi}_1 = (0, \eta, \zeta)^T \in \ker A(0)$. Hence by (4.2) with $\mu = 0$ we have $\zeta = 0$ and therefore $x^+ = (-2/(3\sqrt{3}), x_2, -2/(3\sqrt{3}))^T$. In other words, the jump is parallel to the x_2-axis. A similar conclusion holds for $\tilde{\xi}_2$. The corresponding results for (4.1) are obtained by projections onto the plane $x_3 = 0$ and shown in Figure 4.2.

These results agree fully with those of [SDe] where the (standard) singularly perturbed ODE-form

$$C\dot{x}_1 = -x_2,$$
$$\mu\dot{x}_2 = x_1 - x_2^3 + x_2,$$

is used. This form is obtained by eleminating the variable x_3 from the second differential equation in (4.2) by using the algebraic equation..

As discussed in [SDe], the equivalence of the singular perturbation (4.3) with the perturbation obtained by adding a parasitic inductance in series with the nonlinear resistor rests upon the fact that this resistor is driven by the current x_2. An analogous equivalence pertains in circuits where a small capacitor is added in parallel to a voltage driven resistor (see also Example 3 below). But clearly, when the characteristics of a nonlinear port are not globally solvable in terms of some of the driving variables, then, as noted in [SDe], this is symptomatic of neglected parasitics inside the port whose effects cannot be reflected at the terminals. In this case, we can hardly expect a perturbation by means of the parasitics to be equivalent with a singularly perturbed ODE. Furthermore, no general principle is hidden behind the remark that (4.3) can be derived from (4.1) by formally substituting $\mu\dot{x}_2$ for 0 in the algebraic part of (4.1). To see this, note that (4.1) is unchanged if the equation $0 = x_1 - x_2^3 + x_2$ is replaced by $0 = x_2^3 - x_2 - x_1$ while the system

$$\begin{aligned} C\dot{x}_1 &= -x_2, \\ 0 &= x_2^3 - x_2 - x_1, \end{aligned} \tag{4.3'}$$

is one for which the $\mathcal{P}_\mu$-consistent points of (4.1) are now those corresponding to the *middle* branch of the S-curve in Figure 4.2. That the system (4.3') is physically meaningless is clear from the fact that $\mu\dot{x}_2 = x_2^3 - x_2 - x_1$ amounts to $x_2^3 - x_2 - x_1 = 0$ and $\mu\dot{x}_2 = x_3 - x_1$; that is, the current x_2 flows in the direction opposite to physical reality (see Figure 4.1).

Example 2: The first example was, of course, an idealized circuit and, in practice, the nonlinear, cubic resistor corresponds to a port made up of several resistors and transistors. As an example of such a more realistic version Example 1 we consider the two-transistor port of Figure 4.4a which, as before, is combined with a capacitor into the circuit of Figure 4.4b. The port was discussed in [CYY] and also [TW]. We use as independent variables the voltage x_1 across the port, the current x_2 through the port and the voltage differences $x_3 = v_6 - v_4$, $x_4 = v_3 - v_4$, $x_5 = v_6 - v_1$, $x_5 = v_6 - v_5$, where $v_3, \ldots, v_6$ are the voltage levels at the indicated points of Figure 4.4.

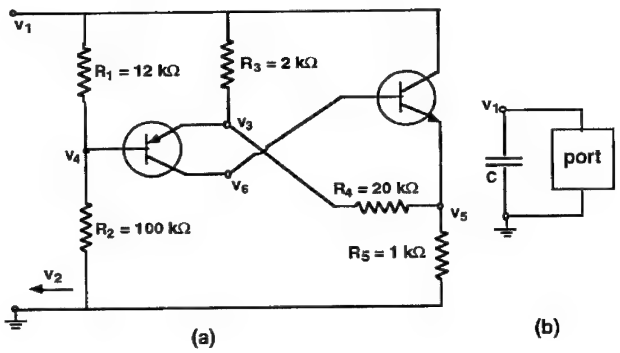

Figure 4.4

Then the circuit is modelled by an autonomous DAE of the form

$$\begin{aligned} C\dot{x}_1 &= x_2, \\ 0 &= Lx + K\Phi(x), \end{aligned} \tag{4.4}$$

where

$$L = \begin{pmatrix} -\frac{1}{R_2} - \frac{1}{R_5} & 1 & \frac{1}{R_2} & 0 & -\frac{1}{R_2} - \frac{1}{R_5} & \frac{1}{R_5} \\ 0 & 0 & \frac{1}{R_3} + \frac{1}{R_4} & -\frac{1}{R_3} - \frac{1}{R_4} & -\frac{1}{R_3} & -\frac{1}{R_4} \\ -\frac{1}{R_2} & 0 & \frac{1}{R_1} + \frac{1}{R_2} & 0 & -\frac{1}{R_1} - \frac{1}{R_2} & 0 \\ -\frac{1}{R_5} & 0 & -\frac{1}{R_4} & \frac{1}{R_4} & -\frac{1}{R_5} & \frac{1}{R_4} + \frac{1}{R_5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(4.5)

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{\alpha_f} & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha_r} - 1 & \frac{1}{\alpha_f} - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \frac{1}{\alpha_f} \\ 0 & 0 & \frac{1}{\alpha_r} & -1 & \frac{1}{\alpha_r} - 1 & \frac{1}{\alpha_f} - 1 \end{pmatrix}, \quad \Phi(u) = \begin{pmatrix} 0 \\ 0 \\ \phi(u_3) \\ \phi(u_4) \\ \phi(u_5) \\ \phi(u_6) \end{pmatrix}.$$

Here the Ebers-Moll transistor model EM1 was applied and we set $\phi(s) = I_{ES}[\exp(qs) - 1]$, where I_{ES} denotes the emitter-base saturation current, and the constant q depends on the temperature. For the transistor constants we used $\alpha_R = 0.5$, $\alpha_F = 0.99$, $I_{ES} = 10^{-16}$, and $q = 40$. For simplicity, in all the calculations we worked with $C = 1$.

The DAE (4.4/5) has a similar behavior as (4.1). The Jacobian of the algebraic part has the form $L + KD\Phi(x)$ where $D\Phi(x) = \text{diag}\,(0, 0, \phi'(x_3), \ldots, \phi'(x_6))$. An evaluation with Mathematica shows that when the sixth colum is deleted the determinant of the resulting 5×5 matrix is a polynomial in the positive quantities $\phi'(x_3), \ldots, \phi'(x_6)$ which has only positive coefficients. Thus this submatrix is nonsingular for all $x \in \mathbb{R}^6$ which implies that the algebraic solution set is a 1-dimensional submanifold of $\mathbb{R}^6$. This path was computed by means of the continuation code PITCON ([BRh]) started from $x = 0$ and its projection onto the x_1, x_2-plane has the S-shaped form of Figure 4.5.

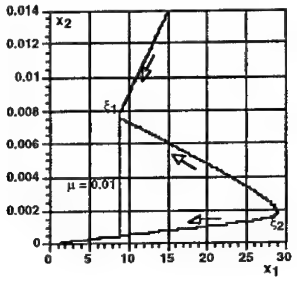

Figure 4.5

Similarly it can be shown that the DAE is geometrically nonsingular of index one: By using the first (linear) algebraic equations x_1 can be we eliminated from the remaining algebraic equations. The resulting system has the form $\Gamma Lx + \Gamma K\Phi(u) = 0$ where Γ is some Gaussian transformation matrix. Now consider the 4×4 matrix obtained by deleting the first row as well as the first and last columns. As before, a Mathematica evaluation shows that the determinant of this matrix is a polynomial in $\phi'(x_3), \ldots, \phi'(x_6)$ which has only negative coefficients. Hence the matrix is nonsingular for all $x \in \mathbb{R}^6$ and the system is nonsingular of index one. On the solution path we observe an accessible as well as an inaccessible singular point with projections ξ_1 and ξ_2, respectively, onto the x_1, x_2-plane.

As before, with a small parasitic inductance $\mu > 0$ in series with the nonlinear port, the resulting circuit, shown in Figure 4.6, is modelled by the DAE

$$(4.6) \qquad C\dot{x}_1 = x_2, \quad \mu\dot{x}_2 = x_1 - x_7, \quad Ly + K\Phi(y) = 0, \quad y = (x_2, \ldots, x_7)^T$$

where L, K, Φ are as in (4.5). Evidently for $\mu = 0$ we obtain an augmented form of (4.4) which is equivalent with it when the identification $x_1 = x_7$ is used. The projection of any solution onto the x_1, x_2-plane is again the path of Figure 4.5.

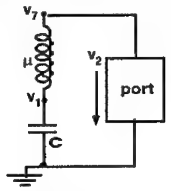

Figure 4.6

For $\mu > 0$ the solutions are constrained to the two-dimensional manifold $W(\mu) = \{x \in \mathbb{R}^7 : Ly + K\Phi(y) = 0\}$, $y = (x_2, \dots, x_7)^T$. As in the case of the unperturbed DAE (4.4) it can be shown that (4.6) is geometrically nonsingular of index one. The matrix representation of (2.9) is here

$$(4.7)\quad \Lambda(\mu, x) = M\,\mathrm{diag}\,(\mu, 0, 0, 0, 0, 0),\ M = \begin{pmatrix} b^T \\ L + KD\Phi(x) \end{pmatrix}^{-1},\ b^T = (1, 0, 0, 0, 0, -1)$$

A computational determination of the single nonzero eigenvalue during the computational trace of the manifold with PITCON showed that, as in the first example, the set of $\mathcal{P}_\mu$ - consistent points, $\mu \in \mathcal{C}_+ = (0, 1)$, consists of all points projecting onto the upper and lower branches of the S-curve shown in Figure 4.5 (excluding, of course the impasse points). Thus we expect only a jump from ξ_1 onto the (unique) point x^- on the lower branch with the same x_1-coordinate. Figure 4.5 already shows an approximating solution for $\mu = 0.01$ computed for (4.6) with the IRK-solver Radau5 of [HW] starting from a point on the upper branch..

A different, no longer physically motivated, perturbation of the original system (4.4/5) is given by the corresponding singularly perturbed ODE

$$\mathrm{diag}\,(C, \mu, \dots, \mu) = \begin{pmatrix} a^T \\ Lx + K\Phi(x) \end{pmatrix},\quad a^T = (0, 1, 0, 0, 0, 0).$$

Thus, in this case, $\Lambda(\mu, x)$ is the inverse of the matrix obtained by deleting the first column of $Lx + KD\Phi(x)$. The eigenvalues of this matrix were computed during a trace

of the solution manifold and, now, it was found that none of the points on the solution manifold is $\mathcal{P}$-consistent. In other words, this singular perturbation provides a different (and erroneous) information about the admissibility or inadmissibility of the solutions of (4.4).

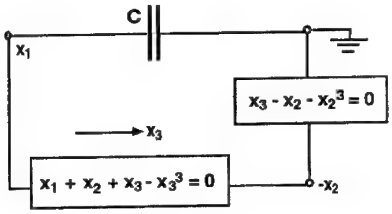

Figure 4.7

Example 3: Earlier we stressed the necessity of allowing the perturbed system to involve a vector-valued μ. As an illustration we consider the circuit of Figure 4.8 which can be modelled by the DAE

$$A\dot{x} = G(x), \quad A = \mathrm{diag}(C,0,0), \quad G(x) = \begin{pmatrix} -x_3 \\ x_1 + x_2 + x_3 - x_3^3 \\ x_3 - x_2 + x_2^3 \end{pmatrix}. \tag{4.8}$$

Thus, in the terminology of [CYY] we have here an S-type and an N-type nonlinear resistors in series. The solutions are constrained to a one-dimensional submanifold of $\mathbb{R}^3$ which has the global parametrization $x = \phi(\sigma) = (\pi(\sigma), \sigma, \sigma - \sigma^3)^T$ where $\pi(\sigma) = -2\sigma + \sigma^3 + (\sigma - \sigma^3)^3$.

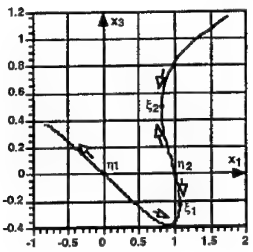

Figure 4.8

Figure 4.8 shows parts of this curve. The system has three equilibria, namely for $\sigma = 0, \pm 1$. Moreover, there are singularities at the points corresponding to zeros of the derivative of π. These are the four points where $\sigma_{\pm}^1 = \pm 0.854398\ldots$, $\sigma_{\pm}^2 = \pm 1.18505\ldots$. In Figure 4.8 the equilibria and singular points on that part of the curve are marked by η_1, η_2 and ξ_1, ξ_2, respectively, and the arrows show the direction of the flow between these points.

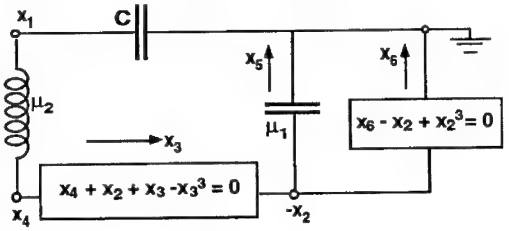

Figure 4.9

Now, in addition to a parasitic inductance in series with the resistors, we introduce, as in [GW], also a small capacitance in parallel with the N-type resistor. The resulting circuit of Figure 4.9 is modelled by the DAE

$$(4.9)\quad A(\mu)\dot{x} = G(x), \quad A(\mu) = \mathrm{diag}\,(C, \mu_1, \mu_2, 0, 0, 0), \quad G(x) = \begin{pmatrix} -x_3 \\ -x_5 \\ x_1 - x_4 \\ x_4 + x_2 + x_3 - x_3^3 \\ x_6 - x_2 + x_2^3 \\ x_3 - x_5 - x_6 \end{pmatrix}.$$

Once again, for $\mu = (\mu_1, \mu_2)^T = 0$ this becomes an augmented version of (4.8) which is equivalent with that DAE under the identification $x_4 = x_1$, $x_5 = 0$, $x_6 = x_3$. Clearly, there is little reason to assume that both μ_1 and μ_2 should be equal. Evidently, the constraint manifold $W(\mu)$ is here a three – dimensional submanifold of $\mathbb{R}^6$ and it is readily verified that (4.9) is completely nonsingular.

A Mathematica calculation shows that for points $\phi(\sigma) \in W(0)$, $\sigma \in \mathbb{R}$, and with $\nu =$

μ_2/μ_1, the nonzero eigenvalues of the operator (2.9) are given by

$$(4.10)\quad \lambda_\pm = -\frac{1}{2\pi'(\sigma)}[q(\sigma,\nu)\pm(q(\sigma,\nu)^2+4\nu\pi'(\sigma))^{1/2}],\ q(\sigma,\nu)=1+\nu(1-3\sigma^2)-(\sigma-\sigma^3)^3.$$

Clearly, the $\mathcal{P}_\mu$-consistent points depend here on the choice of $\mu \in \mathcal{C}_+ = (0,1)^2$. Table 4.1 shows the situation for $\nu = 1$; that is, for $\mu_1 = \mu_2$, and $\sigma < 0$ (coresponding to the part of $W(0)$ in the positive half-plane).

	σ-interval	eigenvalues	
1	$(-0.746664, 0)$	complex	positive real parts
2	$(-0.849913, -0.746664)$	complex	negative real parts
3	$(-0.854398, -0.849913)$	real	both negative
4	$(-1.18505, -0.854398)$	real	one positive, one negative
5	$(-1.33885, -1.18505)$	real	both negative
6	$(-1.46455, -1.33885)$	complex	negative real parts
7	$(-\infty, -1.46455)$	real	both negative

Table 4.1, $\nu = 1$

The table shows that the set $W(0)^{c,\mu}$ of $\mathcal{P}_\mu$ - consistent points where $\mu_1 = \mu_2$ consists of the points $\phi(\sigma) \in W(0)$ with the parameter σ in the intervals numbered 2, 3, 5, 6, 7. Note that at the common endpoint of intervals 1 and 2 a pair of conjugate complex eigenvalues passes through the imaginary axis. Such points were, by explicit assumption, ruled out in [SDe]. Moreover, the $\mathcal{P}_\mu$ - consistent intervals 2, 3 correspond to points of $W(0)$ between the equilibrium η_1 and the singular point ξ_1 on the curve. This shows that it would be fallacious to conclude, say, on the basis of the first two examples, that for problems with a one-dimensional constraint manifold the boundary of the set of $\mathcal{P}_\mu$-consistent points always coincides with the singular points of the unperturbed DAE.

	σ-interval	eigenvalues	
1	$(-0.311604, 0)$	real	both positive
2	$(-0.593199, -0.311604)$	complex	positive real part
3	$(-0.722998, -0.593199)$	complex	negative real part
4	$(-0.854398, -0.722998)$	real	both negative
5	$(-1.18505, -0.854398)$	real	one positive, one negative
6	$(-\infty, -1.18505)$	real	both negative

Table 4.2, $\nu = 10.0$

In order to obtain a picture of the change of $\mathcal{P}_\mu$ - consistency when $\nu = \mu_2/\mu_1$ changes we give in Tables 4.2 and 4.3 the cases $\nu = 10.0$ and $\nu = 0.001$, respectively Note that the $\mathcal{P}_\mu$-consistent intervals 3, 4 corresponding to the points between η_1 and ξ_1 has grown. It it is easily checked that for $\nu \to 0$ the right endpoint of the interval of $\mathcal{P}_\mu$-consistent points tends to $s = 1/\sqrt{3}$.

	σ-interval	eigenvalues	
1	$(-0.854398, 0)$	real	both positive
2	$(-1.18505, -0.854398)$	real	one positive, one negative
3	$(-1.20948, -1.18505)$	real	both positive
4	$(-1.21434, -1.20948)$	complex	positive real part
5	$(-1.22013, -1.21434)$	complex	negative real part
6	$(-\infty, -1.22013)$	real	both negative

Table 4.3, $\nu = 0.001$

Now the interval of $\mathcal{P}_\mu$-consistent points between η_1 and ξ_1 has disappeared and there are two small $\mathcal{P}_\mu$ - consistent intervals 3, 4 corresponding to points on the upper branch of the curve beyond the singular point ξ_2. Moreover, the common point of intervals 4 and 5 is again a point where a pair of complex eigenvalues passes through the imaginary axis.

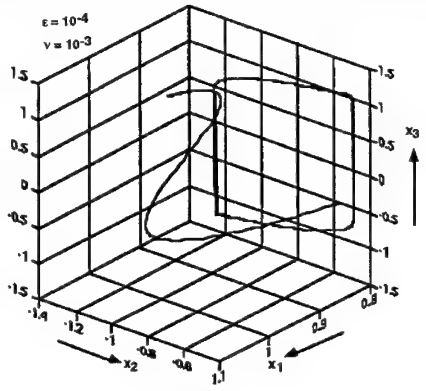

Figure 4.10

It is interesting to analyze the behavior of the solutions of the perturbed problem (4.9) near this point. We expect, of course, a discontinuity. But since in this case there is no

further $\mathcal{P}_\mu$ - consistent point on $W(0)$ in the intersection with ker $A(0)$, there cannot be any jump to another branch. Figure 4.10 shows results of a computation with Radau5 which appear to suggest that the solution of the perturbed system with $\mu_1 = \epsilon = 10^{-4}$ and $\mu_2 = \nu\epsilon$, $\nu = 0.001$ becomes periodic. But this is misleading. In fact the solution is a spiral which slowly tends to infinity. This can be seen well in Figure 4.11 for the case $\nu = 10.0$ (and the same ϵ). In that case, interval 6 in Table 4.2 is $\mathcal{P}_\mu$-consistent and at the singular point ξ_2 the solution becomes the spiral of the figure.

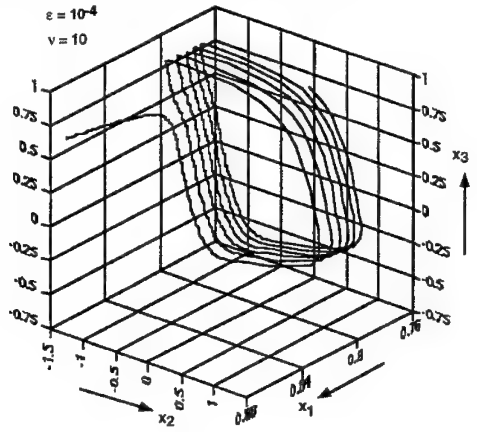

Figure 4.11

The shape of the spiral is easily explained by setting $\mu_1 = \epsilon$, $\mu_2 = \nu\epsilon$ in (4.9). Then by introducing the scaled time $\tau = t/\epsilon$ the differential equations become

$$C\dot{x}_1 = -\epsilon x_3,$$
$$\dot{x}_2 = -x_5,$$
$$\nu\dot{x}_3 = x_1 - x_4.$$

For fixed ν and $\epsilon \to 0$ this shows that x_1 becomes constant in agreement with our theory which requires that the jumps in the discontinuous solution are parallel to ker $A(0)$. With

$x_1 = \gamma = \pi(-1.21463)$ we have then the limiting DAE

$$\begin{aligned}
\dot{x}_2 &= -x_5, \\
\nu\dot{x}_3 &= \gamma - x_4, \\
0 &= x_4 + x_2 + x_3 - x_3^3, \\
0 &= x_6 - x_2 + x_2^3, \\
0 &= x_3 - x_5 - x_6,
\end{aligned}$$

with the perturbation parameter ν. For $\nu = 0$ the solution manifold is one-dimensional and has the form of the cubic $x_2 = -\gamma - x_3 + x_3^3$ when projected into the x_2, x_3-plane. It is readily checked that the upper and lower branches of this cubic are $\mathcal{P}_\nu$-consistent under the given perturbation with $\nu > 0$ and hence that the solution should exhibit jumps of the same type as shown in Figure 4.2. This is indeed confirmed in Figure 4.12 by projecting the results of Figure 4.10 onto the x_2, x_3-plane and superimposing, at the same time, the cubic path. But note that for small positive ν we no longer have $x_4 = \gamma$ and hence that the cubic is slowly transposed toward larger $x_4 = x_1$ values; that is, that the solution is indeed a spiral tending to infinity. This shows that in problems with higher dimensional perturbation vectors the discontinuity behavior may be very different from the straight jumps encountered in the earlier examples.

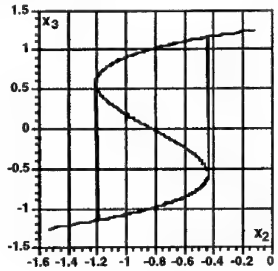

Figure 4.12

Example 4: So far all the examples involved autonomous DAE's. This is, of course, readily changed if, in any of these examples, the dynamics is not only supplied by some

capacitor or inductance, but also by a time-dependent voltage or current source. A simple example of this type is shown in Figure 4.13. It is modelled by a DAE of the form (1.1) with

$$A = \text{diag}\,(L, 0, 0), \quad G(x) = \begin{pmatrix} x_3 - x_2 \\ x_3 - v(t) \\ x_1 + x_2 - x_2^3 \end{pmatrix}, \tag{4.13}$$

where the input voltage v is assumed to be a smooth function of time. Clearly, we have here a two-dimensional constraint manifold W in $\mathbb{R} \times \mathbb{R}^3$. Although, in contrast to the current-driven resistor of the circuit in Figure 4.1, the resistor is now voltage driven, it is evident that the projection of W onto the x_1, x_2-plane, has the form of the cubic path of Figure 4.2. But the impasse points form the submanifolds

$$\{(t, x) \in W : x_1 = \mp 2/(3\sqrt{3}), x_2 = \pm 1/\sqrt{3}\}. \tag{4.14}$$

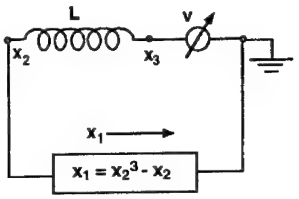

Figure 4.13

In analogy to Figure 4.6 we consider the perturbed circuit of Figure 4.14 with a parasitic capacitor in parallel to the resistor.

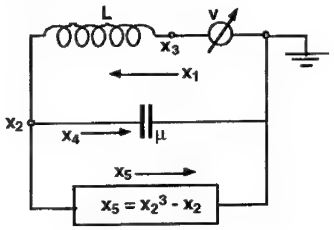

Figure 4.14

This leads to a DAE of the form (4.9) with

$$A(\mu) = \text{diag}\,(L, \mu, 0, 0, 0) \quad G(x) = \begin{pmatrix} x_3 - x_2 \\ x_4 \\ x_3 - v(t) \\ x_5 + x_2 - x_2^3 \\ x_1 - x_4 - x_5 \end{pmatrix}. \tag{4.15}$$

For $\mu = 0$ and with the identification $x_4 = 0$, $x_5 = x_1$ this DAE is equivalent with (1.1)/(4.13). Moreover, for $\mu > 0$, (4.9/14) is easily checked to be geometrically nonsingular of index one. As in the previous examples, the matrix representation of the operator Λ is readily calculated and shows that for $\mu \in \mathcal{C}_+ = (0, 1)$, the set of $\mathcal{P}_\mu$ - consistent points consists of those points $x \in W$ where $|x_2| > \frac{1}{\sqrt{3}}$.

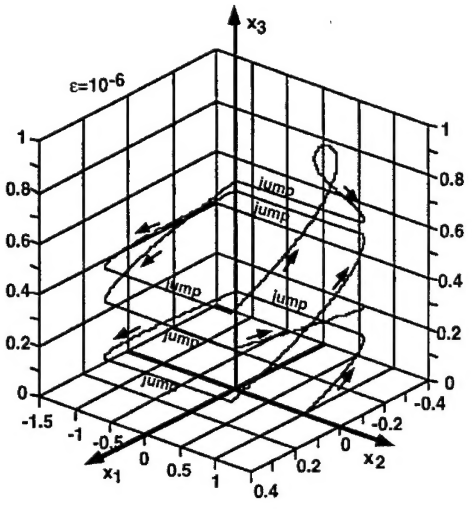

Figure 4.15

A solution of this perturbed DAE (4.9/14) was computed with Radau5 for $L = 1$, $v(t) = \sin(0.2t)$, and the starting point $x = (0, 1, 0, 0, 0)^T$ at time $t = 0$. The result is shown in Figure 4.15. As expected, when the solution reaches a point on the submanifolds

(4.14) of impasse points it jumps between the outside leaves of W. When projected onto the x_1, x_2-plane the picture looks similar to Figure 4.2. But, as Figure 4.15 shows, in x_1, x_2, x_3-space the solution spirals several times until it turns around at a point where $x_2 = x_3$ and then begins a new round of spirals, no longer shown in the figure. This indicates clearly, that for higher dimensional algebraic solution manifolds we may expect a considerably more complex solution behavior than in the one-dimensional case. In fact, it is hardly surprising that in problems of an appropriate form a chaotic regime may be expected to develop as observed by several authors (see e.g. [EC]).

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Part I of this paper presented a theory of distribution solutions of semilinear differential-algebraic equations (DAE's). In particular, it was shown that uniqueness of solutions of initial value problems breaks down completely in the class of discontinuous solutions. Here a mathematical procedure is introduced for selecting physically acceptable solutions which satisfy some new consistency condition relative to admissible perturbations of the original DAE. Several nonlinear circuit examples are given to support the theory.

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